Frequency-domain representation of biosignals

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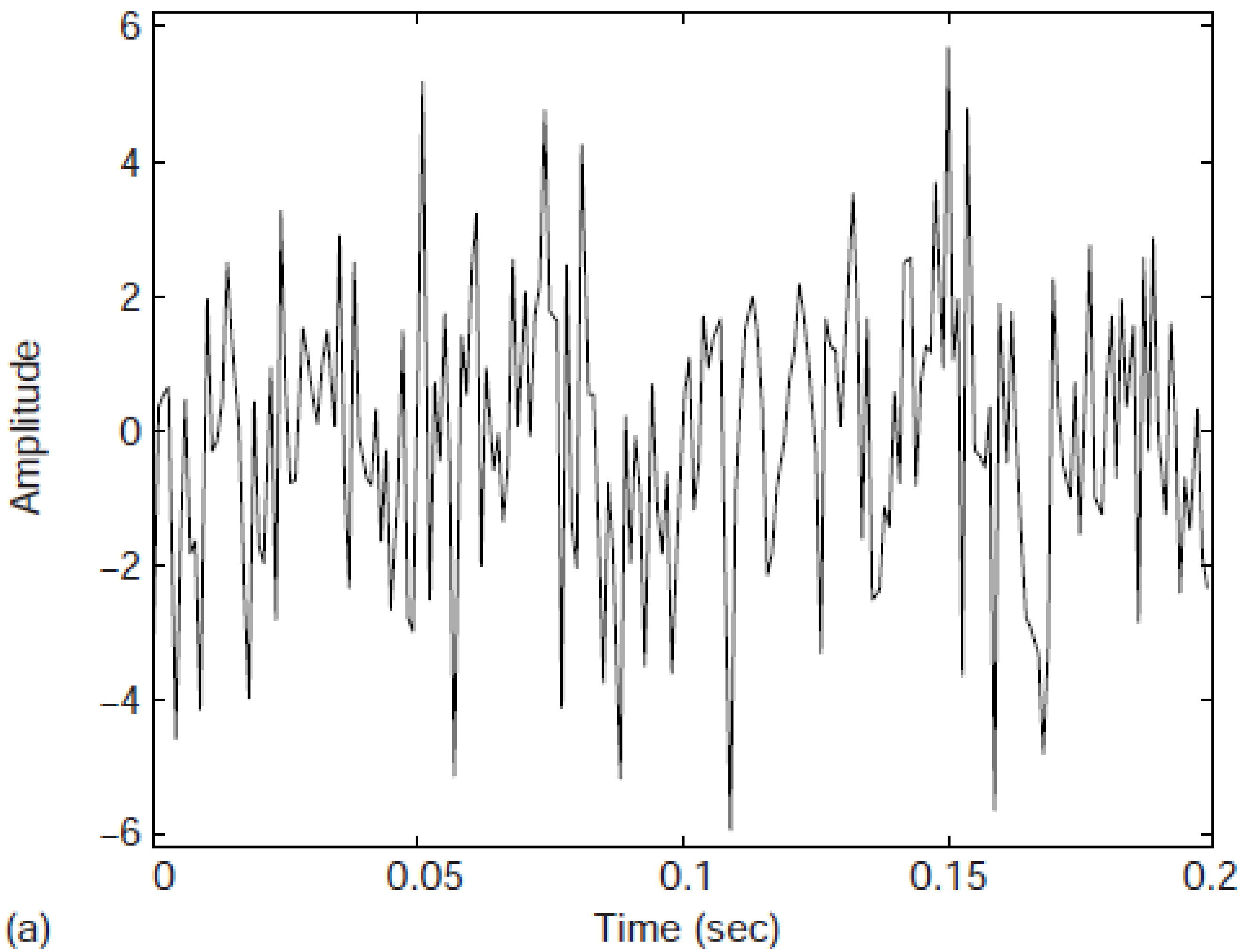
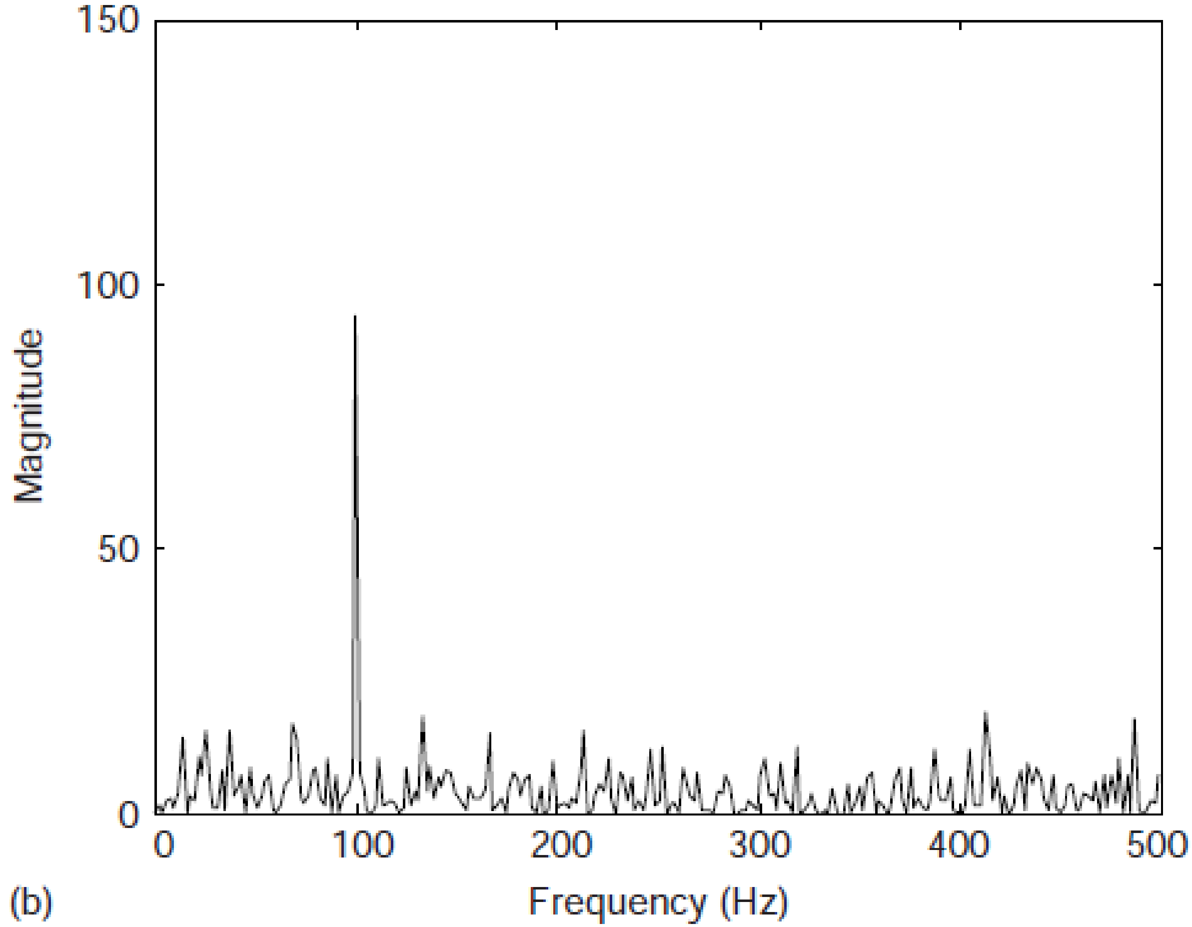
The Fu Foundation School of Engineering and Applied Science

Representation in frequency domain

- Can we see any pattern in this data plot?



Representation in frequency domain



The Fourier series for periodic signals

- A signal is **periodic** if it repeats at fixed intervals.
- Thus, a periodic function of time can be described by a relation of the form

$$f(t) = f(t + kT)$$

where, $k = \ldots, -2, -1, 0, 1, 2, \ldots$.

The Fourier series

- It is possible to represent a periodic signal, **f(t)** with period **T**, by an infinite series of sinusoids, called the Fourier series. The trigonometric Fourier series is defined by:

$$x(t) = a_0 + \sum_{m=1}^{\infty} (a_m \cos m\omega_o t + b_m \sin m\omega_o t)$$

where $\omega_0 = 2\pi/T$.

The Fourier series

- The coefficients of the Fourier series are determined as

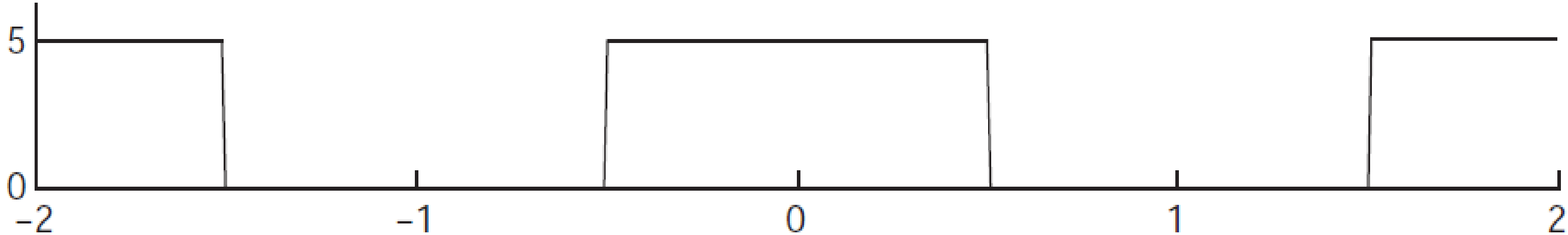
$$a_0 = \frac{1}{\mathrm{T}} \int_{\mathrm{T}} x(t) dt$$

$$a_m = \frac{2}{\mathrm{T}} \int_{\mathrm{T}} x(t) \cos(m\omega_o t) dt$$

$$b_m = \frac{2}{\mathrm{T}} \int_{\mathrm{T}} x(t) \sin(m\omega_o t) dt$$

Example

- Find the Fourier series for the following periodic function with T = 2



Example

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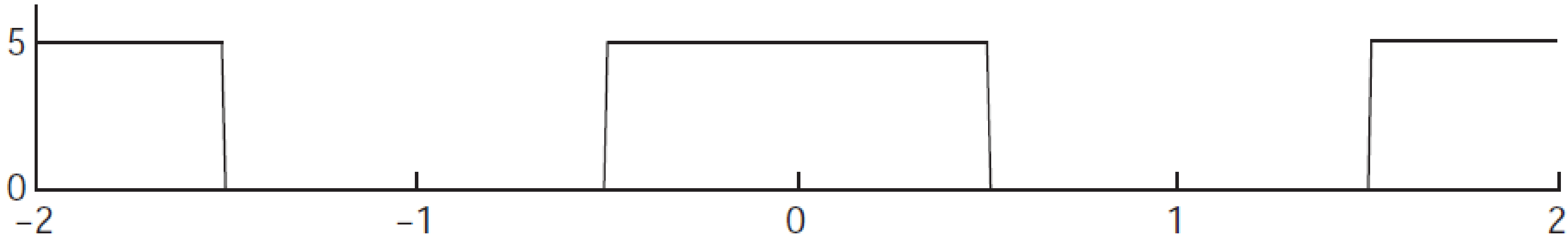
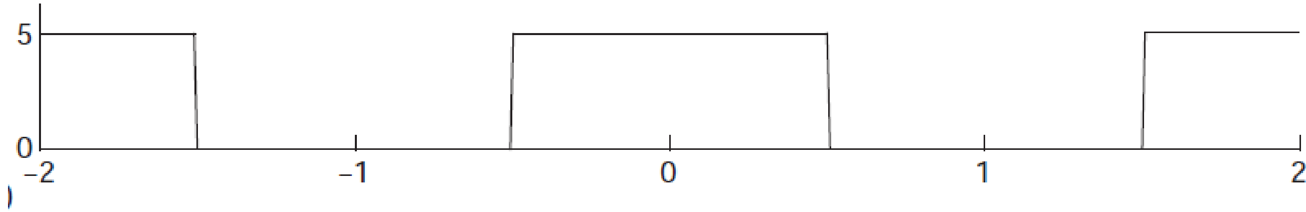


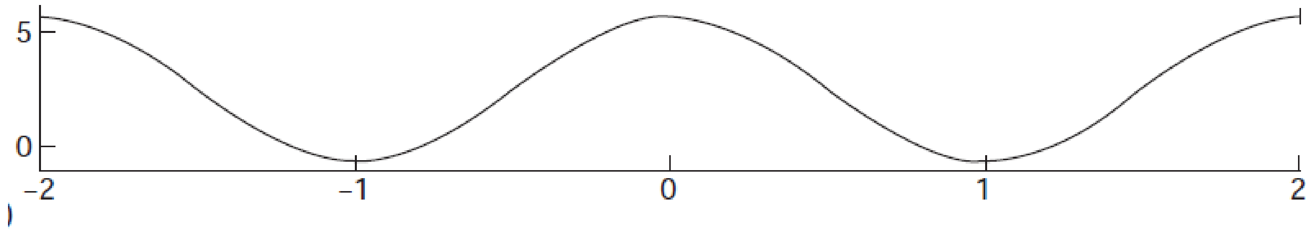
$$x(t) = 2.5 + 10 \cdot \sum_{m=1}^{\infty} \frac{\sin(m\pi/2)}{m\pi} \cos(m\pi t)$$

The Fourier series

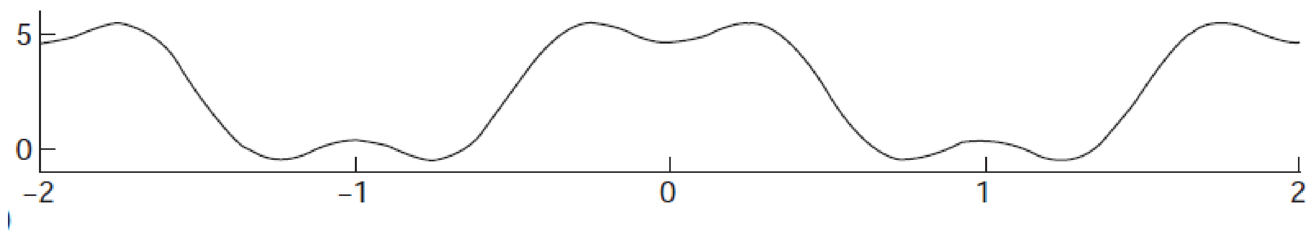
Original signal:



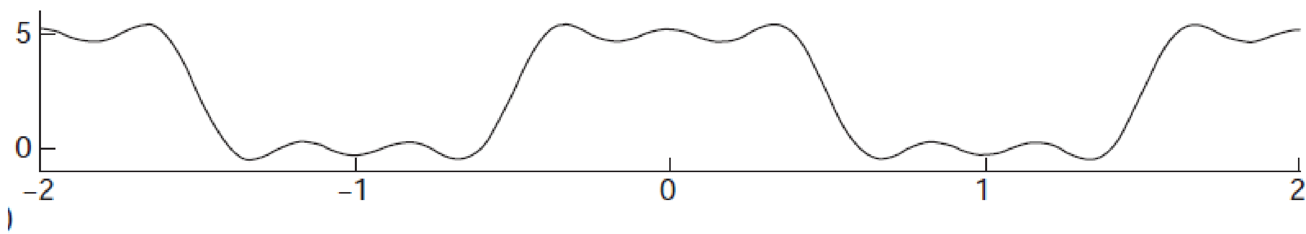
$$a_0 + (a_1 \cos(\varpi_0 t) + b_1 \sin(\varpi_0 t))$$



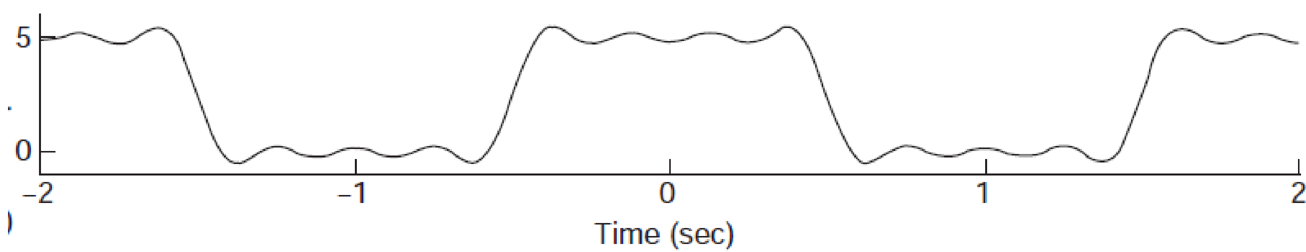
$$a_0 + \sum_{m=1}^{3} (a_m \cos(m\varpi_0 t) + b_m \sin(m\varpi_0 t))$$



$$a_0 + \sum_{m=1}^{5} (a_m \cos(m\varpi_0 t) + b_m \sin(m\varpi_0 t))$$



$$a_0 + \sum_{m=1}^{7} (a_m \cos(m\varpi_0 t) + b_m \sin(m\varpi_0 t))$$



The Fourier series

- Fourier series tells us that the periodic signal, x(t), is precisely replicated by summing an infinite number of sinusoids
- $\omega_0 = 2\pi/T$ is the fundamental frequency of x(t)
- The frequencies of the sinusoid functions always occur at integer multiples of ω_0 and are referred to as “harmonics” of the fundamental frequency
- In practice, many periodic or quasi-periodic biological signals can be accurately approximated with only a few harmonic components

Compact Fourier Series

Original form:

$$x(t) = a_0 + \sum_{m=1}^{\infty} (a_m \cos m\omega_o t + b_m \sin m\omega_o t)$$

Since $$a_m \cos m\omega_o t + b_m \sin m\omega_o t = A_m \cos(m\omega_o t + \phi_m)$$

Compact form:

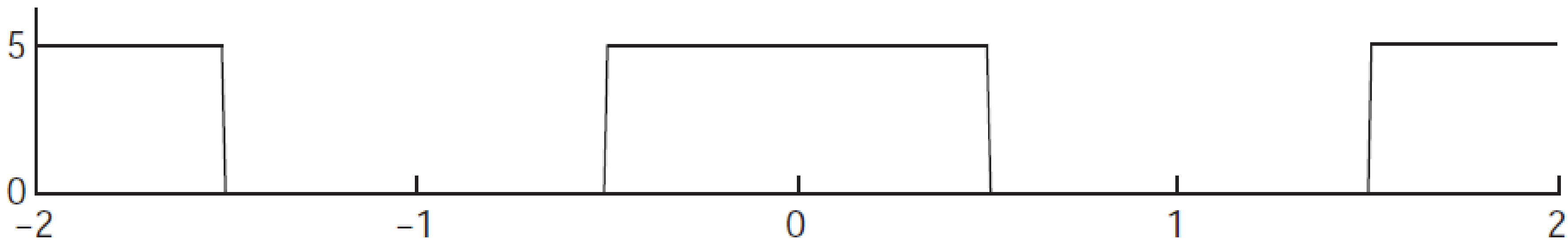
$$x(t) = A_0 + \sum_{m=1}^{\infty} A_m \cos(m\omega_o t + \phi_m).$$

Where: $$A_m = \sqrt{a_m^2 + b_m^2}$$

$$\phi_m = \tan^{-1}\left(\frac{-b_m}{a_m}\right)$$

The compact Fourier series

- Find the compact Fourier series for the following periodic function with T = 2



$$A_m = |a_m| = \frac{10}{m\pi}\left|\sin\left(\frac{m\pi}{2}\right)\right| = \begin{cases} \dfrac{10}{m\pi} & m = 1,3,5\ldots \\ 0 & m = 2,4,6\ldots \end{cases}$$

$$\Phi_m = \tan^{-1}\left(\frac{-b_m}{a_m}\right) = \tan^{-1}\left(\frac{0}{5\sin(m\pi/2)/m\pi}\right) = \begin{cases} 0 & m = 0,1,4,5\ldots \\ \pi & m = 2,3,6,7\ldots \end{cases}$$

The Fourier series

- A periodic signal can be expressed either in the "time-domain" by the signal's time function, x(t), or in the "frequency-domain" by specifying the Fourier coefficient and phase, A_m and ϕ_m, as a function of the signal's harmonic frequencies, $m\omega_0$.
- If we know the Fourier coefficients and the frequency components that make up the signal, we can fully recover the periodic signal x(t).

Fourier transform for aperiodic signals

- One of the disadvantages of the Fourier series is that it applies only to periodic signals
- Many biological signals are continuous functions of time, and never repeat in time
- The Fourier transform, is used to decompose a continuous aperiodic signal into its constituent frequency components

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier transform

- $X(\omega)$ is a complex valued function of the continuous frequency, ω.
- $X(\omega)$ can be transformed back to time domain using the inverse Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

Discrete Fourier Transform

- DFT is necessary to analyze and represent discrete signals in the frequency domain.

$$X(m) = \sum_{k=0}^{N-1} x(k) e^{-j\frac{2\pi mk}{N}}; m = 0, 1, \ldots, N-1$$

Where the index m represents the digital frequency index, x(k) is the sampled data of x(t), and N is an even number that represents the number of samples for x(k)

Fast Fourier Transform (FFT)

- DFT needs $O(N^2)$ operations because it has N/2 outputs X_k, and each output requires a sum of N terms
- **If N is a power of 2**, some tricks can be used compute the same result with only $(N/2)\log_2(N)$ operations. This is called Fast Fourier Transform (FFT), which is significantly faster.

Estimation issues

- The highest frequency that can be estimated is determined by the rate at which the signal was sampled and is given by

$$f_{\max} = \frac{1}{2T_s} = \frac{f_s}{2}$$

Where T_s is sampling interval

Estimation issues

- The frequency resolution of spectral estimates is also determined by the segment length and the sampling frequency

$$\Delta f = \frac{f_s}{N}$$

Where f_s is sampling frequency, and N is the length of data

Estimation issues

- As with the Fourier transform, record length selection is particularly important when computing the spectra of periodic signals; it is best to choose a record length that yields a frequency increment that divides evenly into the signal's fundamental frequency.
- For FFT, since the number of samples must be a power of 2, it sometimes may lead to estimates at strange frequencies.

Demo in the class

```
dt=0.01; %fs=100Hz
t=0:dt:10-dt;
y1=sin(2*pi*10.15*t);
%y1 = sin(2*pi*9.8*t);
N = length(y1); % # of data points: 1000
y_fft=abs(fft(y1))/(N/2);
df = 1/dt/N;
f=(0:1:length(t)-1)*df;
plot(f,y_fft,'-s');
xlabel('Frequency (Hz)');
ylabel('Amplitude');
```